Defining Mass for Black Holes with Scalar Field Hair in anti-de Sitter Spacetime

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Outline

- Gravity coupled to matter has been studied since 1970s
- No hair conjecture: only fields with geometrical Gauss-like law can exist in exterior region of BH, scalar hair excluded
- NH theorems require asymptotic flatness and convex potential
- Stable solutions found in asymptotically AdS with non-convex potential
- Mass of these spacetimes are required to study thermodynamical properties
- Our work: complete the theory for scalar field minimally coupled to gravity by calculating the mass and study the thermodynamics for all possible cases

Spacetime

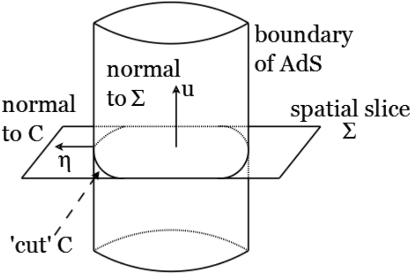
- If there are Killing vectors in space-time, symmetries exist
- Noether's theorem: to every symmetry corresponds a conserved charge
- Conservation of mass associated to a time-like Killing vector ξ^μ (time translation invariance)
- The spacetime is d-dimensional AdS with deviation:

$$ds^2 = -\left(k + \frac{r^2}{L^2}\right) dt^2 + \frac{dr^2}{k + \frac{r^2}{L^2}} + r^2\, d\sigma^2_{d-2,k} + h_{\mu\nu}\, dx^\mu\, dx^\nu$$

Henneaux method

Henneaux's mass is defined as:

$$Q_0 = \frac{1}{16\pi G}\int_C G_a{}^{bdc}[\xi^e \hat{u}_e D_b h_{cd} - h_{cd} D_b(\xi^e \hat{u}_e)]\hat{\eta}^a dS + \frac{1}{4\pi G}\int_C (\kappa_{ab} - \kappa q_{ab})\xi^a \hat{\eta}^b dS.$$



Boundary conditions in the absence of matter field

- For matter-free gravity the asymptotic behaviour of the metric is:

$$h_{rr} = O(r^{-d-1}), \quad h_{mn} = O(r^{-d+3})$$

where m, n include the time coordinate t and the $(d-2)$ angles.

- These boundary conditions guarantee the masses to be finite.
- They are satisfied by AdS-Schwarzschild in 4-d:

$$ds^2 = -\left(1 + \frac{r^2}{L^2} - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{r^2}{L^2} - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

where $L = \sqrt{-\frac{3}{\Lambda}}$ is the radius of curvature

- The mass is $Q_0 = M$

Action for matter coupled to gravity

We consider the action

$$I[g,\phi] = \int d^d x \sqrt{-g}\left[\frac{R-2\Lambda}{16\pi G} - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right]$$

where the scalar field $\phi(r)$ depends on r and has a potential $V(\phi)$. We consider the metric ansatz:

$$ds^2 = -H(r)e^{2\delta(r)}dt^2 + H(r)^{-1}dr^2 + r^2\, d\sigma^2_{d-2,k}$$

where $H(r)$ and $\delta(r)$ are metric functions and

$$d\sigma^2_{d-2,k} = d\psi^2 + f_k^2(\psi)\, d\Omega^2$$

is the line element of a $(d-2)$-dimensional horizon with constant curvature.

Topological black holes and field equations

The function $f_k(\psi)$ depends on k as follows:

$$f_k(\psi) = \begin{cases} \sin\psi & \text{for } k = 1 \\ \psi & \text{for } k = 0 \\ \sinh\psi & \text{for } k = -1. \end{cases}$$

The field equations for AdS minimally coupled to a scalar field are:

$$\begin{aligned} 0 =& H\phi'' + \left[H' + H\delta' + (d-2)\frac{H}{r}\right]\phi' - \frac{\partial V}{\partial \phi} \\ 0 =& \frac{d-2}{2r}\left[H' + (H-k)\frac{(d-3)}{r}\right] + \frac{1}{2}H\phi'^2 + \Lambda + V(\phi) \\ 0 =& (d-2)\frac{\delta'}{r} - \phi'^2 \end{aligned}$$

Scalar field asymptotics

- The scalar field has the form: $\phi \sim r^{-\Delta}$
- From the field equations we obtain:

$$\frac{1}{L^2}\Delta(\Delta+1) - \frac{d}{L^2}\Delta - m^2 = 0$$

- Solving for Δ we have:

$$\Delta_{\pm} = \frac{(d-1)}{2}\left[1 \pm \sqrt{1 + \frac{4m^2L^2}{(d-1)^2}}\right]$$

where $m^2 = \left.\frac{\partial^2 V(\phi)}{\partial \phi^2}\right|_{\phi=0}$ is not necessarily positive. We refer to m as the 'mass' of the scalar field.

Summary table

We can write the scalar field in a more general form:

$$\phi(r) = ar^{\Delta_-} + br^{\Delta_+}$$

case	$\Delta_\pm$	$\phi(r)$
case 1	$\Delta_+ > 0, \Delta_- < 0$ $\Delta_+, \Delta_- \in \mathbb{R}$	$ar^{-\Delta_+} + \dots$
case 2	$\Delta_+ > 0, \Delta_- > 0$ $\Delta_+, \Delta_- \in \mathbb{R}$	$ar^{-\Delta_+} + \dots + br^{-\Delta_-} + \dots$
case 3	$\Delta_+ > 0, \Delta_- > 0$ $\Delta_+ - \Delta_- \in \mathbb{Z}^+$	$ar^{-\Delta_+} \dots + br^{-\Delta_-} + \dots + c\ln(r)r^{-\Delta_+}$
case 4	$\Delta_+ = \overline{\Delta}_-$	$ar^{\delta}\cos(\sigma\ln(r))$

No hair results

To prove the non-existence of hair for the cases we:

- Multiply our field equations by $\phi r^{d-2} e^{\delta}$ and integrate form $x = r_h$ to $y = \infty$:

$$\int_x^y dr\, r^{d-2} e^{\delta} \left(\phi \frac{\partial V}{\partial \phi} + H\phi'^2 \right) - \left[H\phi\phi' r^{d-2} e^{\delta} \right]_x^y = 0$$

- We assume convex potential with $\phi \frac{\partial V}{\partial \phi} > 0$ and we assume $H > 0$
- Example: potentials of the form $V = \frac{1}{2} m^2 \phi^2$ and $V = \lambda \phi^4$

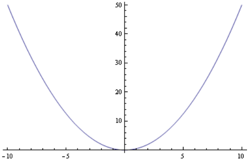


Figure: Convex potential

No hair results

$$\int_x^y dr\, r^{d-2} e^{\delta} \left(\phi \frac{\partial V}{\partial \phi} + H\phi'^2 \right) - \left[H\phi\phi' r^{d-2} e^{\delta} \right]_x^y = 0$$

What happens to the boundary term at infinity?

- Boundary term must be positive, so $\phi\phi' > 0$, if not, mathematical contradiction $\rightarrow$no hair
- If boundary term cancels, the integral is zero, $\rightarrow$no hair
- We have proven that the boundary term vanishes for all cases if the potential is convex
- For non-convex potentials like the Higgs potential $\frac{\lambda}{4}(\phi^2 - v^2)$ there is scalar hair.

Matter coupled to gravity

- When a scalar field is added the asymptotic conditions on the metric are no longer satisfied
- Scalar field has slow fall off at ∞ which has a back reaction on metric and modifies the asymptotic behaviour of the metric
- Q_0 is no longer finite
- The solution to make the divergences cancel is to add a scalar field contribution to the mass
- The total mass of the spacetime is:

$$Q(\xi) = Q_G(\xi) + Q_\phi(\xi)$$

- The gravitational contribution is

$$Q_G = Q_0 + \Delta Q$$

where Q_0 is the gravitational contribution defined previously
- The extra contribution ΔQ is a non-linear correction in the deviation from the background metric.

$$\Delta Q = -12\pi G \int \frac{r^6}{L^5} \xi h_{rr}^2 dS$$

- The scalar contribution is of the form:

$$Q_\phi(\xi) = \frac{1}{6L} \int r^2 \xi_t u^t [L^2 (n^r \partial_r \phi)^2 - m^2 L^2 \phi^2 + k_3 \phi^3 + k_4 \phi^4 + k_5 \phi^5] \, dS$$

- Both Q_G and Q_ϕ are divergent, we have to include the right number of terms in each expression to make divergences cancel.

- To find finite expressions of mass we consider subleading terms in expansions of the scalar field and metric perturbation
- We consider expansions of the form:

$$\phi(r) = ar^{-\Delta_-} + \beta_1 a^2 r^{-2\Delta_-} + ... + br^{-\Delta_+} + ...$$

- In our expansions we want to include both solutions Δ_- and Δ_+
- Since $\Delta_- < \Delta_+$ we have to find all the terms with powers $n\Delta_-$ which are larger than $r^{-\Delta_+}$
- All the other terms can be ignored
- We obtain several subcases; the number of terms depend on the values m, Δ_- and Δ_+
- The expressions for the metric deviation h_{rr} are also obtained from the field equations

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Example in 4 dimensions: $-\frac{27}{16L^2} < m^2 < -\frac{9}{4L^2} + \frac{81}{100L^2}$

- From the field equations we have the expression for the asymptotics for the scalar field

$$\phi(r) = ar^{-\Delta_-} + \beta_1 a^2 r^{-2\Delta_-} + \beta_2 a^3 r^{-3\Delta_-} + br^{-\Delta_+} + \ldots$$

- The expression for the deviation is:

$$h_{rr} = \frac{L^2}{16\pi G r^2}(\alpha_1 a^2 r^{-2\Delta_-} + \alpha_2 a^3 r^{-3\Delta_-} + \alpha_3 a^4 r^{-4\Delta_-}) + \frac{f_{rr}}{r^5} + \ldots$$

- For this example the mass is

$$Q(\xi) = \frac{16\pi G\, f_{rr}}{L^2} + ab\left(\frac{3}{4L^2} - \frac{m^2}{3}\right)$$

- We are interested in scalar field matter minimally coupled to AdS gravity
- Found field equations for gravity minimally coupled to scalar hair
- We have found the four possible asymptotics near ∞
- Proven no-hair theorems for convex potential
- We constructed the masses by calculating gravitational and scalar contribution and making divergences cancel
- Next we will produce numerical results for the scalar field and metric functions
- We will study the thermodynamics of black hole solutions

Thank you.