On the existence of dyons and dyonic black holes in Einstein-Yang-Mills theory

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CQG **29** (2012) 235024

Outline

- Soliton and black hole equilibrium solutions in Einstein-Yang-Mills (EYM) theory.
- Dyonic solutions in asymptotically anti-de Sitter (aAdS) spacetimes.
- Local existence at the regular centre/horizon/infinity.
- Global existence via the implicit function theorem.
- Conclusions and further work.

Black holes and solitons in asymptotically flat EYM theory.

- EYM theory: General Relativity with matter described by a classical gauge field satisfying the Yang-Mills equations.
- Bartnik & McKinnon (1988): globally regular equilibrium solutions (solitons) of the EYM equations in 4-dim asymptotically flat, spherically symmetric spacetime; gauge group $\mathfrak{su}(2)$.
- Corresponding BH solutions found soon after, but both classes were found to be unstable (Volkov and Gal'tsov 1999).
- Gauge fields purely magnetic.

Asymptotically anti-de Sitter (aAdS) spacetime.

- Change asymptotics to asymptotically anti-de Sitter: black hole (Winstanley 1999) and soliton (Bjoraker & Hosotani 2000) solutions found with purely magnetic gauge fields.
- Among these are solutions stable to linear perturbations.
- Similar results hold with the gauge group enlarged to $\mathfrak{su}(N)$ (Shepherd & Winstanley 2012).
- Numerical work of Bjoraker & Hosotani: solutions with non-trivial electric and magnetic components of the gauge field: **dyons**.
- Contrast the asymptotically flat solutions: only solution with non-zero electric part is Reissner-Nordström (Ershov and Gal'tsov 1989).

Questions

- Can we prove existence of dyons and dyonic black holes?
  - static spherically symmetric EYM;
  - $\mathfrak{su}(2)$ gauge field with non-trivial electric and magnetic components;
  - anti-de Sitter asymptotics.
- Are these solutions stable?

$\mathfrak{su}(2)$ EYM theory in aAdS spacetime

- Four-dimensional EYM theory with gauge group $\mathfrak{su}(2)$ and a negative cosmological constant Λ:

$$S_{\mathrm{EYM}} = \frac{1}{2}\int d^4x\sqrt{-g}\,[R - 2\Lambda - \mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}]\,.$$

- Vary the action to obtain

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} &= T_{\mu\nu}, \\ \nabla_\mu F_\nu{}^\mu + [A_\mu, F_\nu{}^\mu] &= 0. \end{aligned}$$

The line element

- We consider static, spherically symmetric spacetimes on which asymptotically anti-de Sitter boundary conditions will be imposed.

$$ds^2 = -\mu(r)S(r)^2dt^2 + \mu(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

- Introduce the mass function $m(r)$ by

$$\mu(r) = 1 - \frac{2m(r)}{r} + \frac{r^2}{\ell^2},$$

where ℓ is the AdS radius of curvature defined by

$$\ell^2 = -\frac{3}{\Lambda}.$$

The gauge field

- Gauge field $F_{\mu\nu}$ is given in terms of the gauge potential A_μ (an $\mathfrak{su}(2)-$valued 1-form) by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

- The potential is given by

$$\mathbf{A} = \mathcal{A}\, dt + \frac{1}{2}\left(C - C^H\right) d\theta - \frac{i}{2}\left[\left(C + C^H\right)\sin\theta + D\cos\theta\right] d\varphi,$$

where $\mathcal{A}$, C and D are 2×2 matrices, given by

$$\mathcal{A} = \frac{i}{2}\begin{pmatrix} \alpha(r) & 0 \\ 0 & -\alpha(r) \end{pmatrix}, C = \begin{pmatrix} 0 & \omega(r) \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with $\alpha(r)$ and $\omega(r)$ being real functions of r only.

- The functions α and ω are respectively the electric and magnetic gauge fields.

Field equations

- These reduce to two Einstein equations...

$$
\begin{aligned}
m' &= \frac{r^2\alpha'^2}{2S^2} + \frac{\alpha^2\omega^2}{\mu S^2} + \mu\omega'^2 + \frac{\left(\omega^2-1\right)^2}{2r^2}, \\
\frac{S'}{S} &= \frac{2\alpha^2\omega^2}{r\mu^2 S^2} + \frac{2\omega'^2}{r},
\end{aligned}
$$

- ...and two Yang-Mills equations:

$$
\begin{aligned}
\alpha'' &= -\frac{2\alpha'}{r} + \frac{\alpha' S'}{S} + \frac{2\alpha\omega^2}{r^2\mu}, \\
\omega'' &= -\frac{\omega' S'}{S} - \frac{\omega'\mu'}{\mu} - \frac{\alpha^2\omega}{\mu^2 S^2} + \frac{\omega\left(\omega^2-1\right)}{r^2\mu}.
\end{aligned}
$$

- Notice that there are singular points at $r=0$, $r=\infty$ and $r=r_h$ where $\mu(r_h)=0$ - corresponding to an event horizon. [$S=0$ cannot arise.]

Boundary conditions: regular origin.

- Solutions with a regular origin must have

$$\begin{aligned} m(r) &= m_3 r^3 + O(r^4), \\ S(r) &= S_0 + S_1 r^2 + O(r^3), \\ \alpha(r) &= \alpha_1 r + \alpha_3 r^3 + O(r^4), \\ \omega(r) &= 1 + \omega_2 r^2 + O(r^3) \end{aligned}$$

  with m_3, S_1, α_3 given explicitly in terms of S_0, α_1, ω_2.
- Anticipate a three-parameter family of solutions with a regular origin (four parameter including ℓ).

Boundary conditions: event horizon.

- For a regular, non-extremal BH event horizon at $r = r_h > 0$, we have $\mu(r_h) = 0, \mu'(r_h) > 0$.

$$
\begin{aligned}
m(r) &= \frac{r_h}{2} + \frac{r_h^3}{2\ell^2} + m'_h\,(r - r_h) + O(r - r_h)^2, \\
S(r) &= S_h + S'_h\,(r - r_h) + O(r - r_h)^2, \\
\alpha(r) &= \alpha'_h\,(r - r_h) + O(r - r_h)^2, \\
\omega(r) &= \omega_h + \omega'_h\,(r - r_h) + O(r - r_h)^2.
\end{aligned}
$$

- The free parameters are $r_h, S_h, \alpha'_h, \omega_h$.
- Refer to these as the event horizon asymptotic conditions.

Boundary conditions: infinity.

- Asymptotically anti-de Sitter boundary conditions are encoded in $S(r) \to 1, m(r) \to M$ (constant) as $r \to \infty$.

$$
\begin{aligned}
m(r) &= M - \frac{1}{r}\left(\frac{d_1^2}{2} + \alpha_\infty^2 \omega_\infty^2 \ell^2 + \frac{c_1^2}{\ell^2} + \frac{\left(\omega_\infty^2 - 1\right)^2}{2}\right) + O(r^{-2}), \\
S(r) &= 1 - \frac{1}{2r^4}\left(\alpha_\infty^2 \omega_\infty^2 \ell^4 + c_1^2\right) + O(r^{-5}), \\
\alpha(r) &= \alpha_\infty + \frac{d_1}{r} + O(r^{-2}), \\
\omega(r) &= \omega_\infty + \frac{c_1}{r} + O(r^{-2}).
\end{aligned}
$$

- The field equations place no restrictions on the constants α_∞, d_1, ω_∞, c_1 or M, giving an expected five-parameter family of solutions.
- Refer to these as the asymptotically AdS boundary conditions.

Trivial solutions

We note the following trivial solutions of the $\mathfrak{su}(2)$ EYM equations:

(i) Set $\alpha = 0, \omega = \pm 1$. This yields the Schwarzschild-AdS black hole solution.

(ii) Set $\alpha = \omega = 0$. Then S is constant and $m(r) = M - 1/2r$. This yields the magnetically charged Abelian RN-AdS black hole with $Q_{\mathrm{M}} = 1$.

(iii) Set $\omega = 0, \alpha = Q_{\mathrm{E}}/r$. This yields the Abelian RN-AdS black hole with charge $Q_{\mathrm{E}}, Q_{\mathrm{M}} = 1$.

(iv) Set $\alpha = 0$ to obtain the equations for purely magnetic solutions: these have been studied in depth (Winstanley 2009).

Global solution strategy

- Apply local existence results at the event horizon (regular centre) and at infinity.
- These solutions are analytic in the free parameters: solution set includes the trivial solution (i) above (Schwarzschild-AdS).
- Apply nonlinear perturbation theory (implicit function theorem) to show that there is a three-dimensional manifold of solutions that form a neighbourhood of the trivial solution in the appropriate function space, and that ‘glues’ the local solutions to form global solutions.
- Some details below for the black hole case: the soliton case is similar.

Local existence: the BFM theorem

Due to the singularities in the equations, standard existence results do not apply. The following theorem (Breitenlohner, Forgacs & Maison 1994) can be applied.

Theorem

Given $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{v} \in \mathbb{R}^m$, *constants* $\tau_i > 0$ *and integers* $\sigma_i, \varsigma_i \geq 1$.

$$x\frac{du_i}{dx} = x^{\sigma_i} f_i\left(x, \mathbf{u}, \mathbf{v}\right),$$
$$x\frac{dv_i}{dx} = -\tau_i v_i + x^{\varsigma_i} g_i\left(x, \mathbf{u}, \mathbf{v}\right), \qquad (1)$$

f_i, g_i *analytic in a neighbourhood of* $x = 0$, $\mathbf{u} = \mathbf{c}$, $\mathbf{v} = \mathbf{0}$ *for all* $\mathbf{c} \in \mathcal{C} \subseteq \mathbb{R}^n$ *(open). Then there is an n-param soln of (1) with*

$$u_i(x) = c_i + O\left(x^{\sigma_i}\right), \qquad v_i(x) = O\left(x^{\varsigma_i}\right), \qquad (2)$$

where $u_i(x), v_i(x)$, *are defined for* $\mathbf{c} \in \mathcal{C}$, *for* $|x| < x_0(\mathbf{c})$ *(for some* $x_0(\mathbf{c}) > 0$) *and are analytic in* x *and* $\mathbf{c}$.

Local existence results

Proposition

(Local existence of solutions in a neighbourhood of the event horizon)
There exists a four-parameter family of local solutions of the EYM field equations near an event horizon at $r = r_h$, satisfying the boundary conditions given above for solutions with a regular event horizon. These solutions are analytic in r_h, S_h, α'_h, ω_h, r and ℓ.

Proposition

(Local existence of solutions in a neighbourhood of infinity)
There exists a five-parameter family of local solutions of the EYM field equations near $r \to \infty$, satisfying the boundary conditions above for AdS asymptotics. These solutions are analytic in M, α_∞, ω_∞, c_1, d_1, r^{-1} and ℓ.

Proofs: Rewrite the EYM equations in a form in which the BFM theorem can be applied.

Implicit function theorem

Theorem

If $f(c_0, y_0) = 0$ *and* $\frac{\partial f}{\partial y}$ *is non-zero at* (c_0, y_0)*, then we can solve the equation* $f(c, y) = 0$ *in a neighbourhood of* (c_0, y_0)*, and represent the solution with a continuous function* $y = g(c)$*.*

Example

The equation $f(c, y) = c^2 \sin y + y^2 \cos c - 1 = 0$ has the solution $(c_0, y_0) = (0, 1)$. Noting that $\frac{\partial f}{\partial y}(0, 1) = 2$, we can apply the theorem to prove that there exists a continuous function g and a neighbourhood I of $c_0 = 0$ such that

$$y = g(c) \Leftrightarrow c^2 \sin y + y^2 \cos c - 1 = 0, \quad c \in I.$$

Implicit function theorem (multivariable calculus version)

Theorem

Let C, Y, Z be open subsets of $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^m$ respectively. Let U be a neighbourhood of $\mathbf{c_0}$ in C, V be a neighbourhood of $\mathbf{y_0}$ in Y and let $f : U \times V \to Z$ be a C^1 mapping:

$$f : (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{z} = f(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m.$$

Suppose that $f(\mathbf{c_0}, \mathbf{y_0}) = 0$ and that the matrix $\frac{\partial f_i}{\partial y_j}(\mathbf{c_0}, \mathbf{y_0})$ is invertible. Then there is a neighbourhood $\tilde{U}$ of $\mathbf{c_0}$ in C, a neighbourhood $\tilde{V}$ of $\mathbf{y_0}$ in Y and a unique continuous mapping $g : \tilde{U} \to \tilde{V}$ such that $\mathbf{y_0} = g(\mathbf{c_0})$ and $f(\mathbf{c}, G(\mathbf{c})) = 0$ for all $x \in \tilde{U}$.

Implicit function theorem (Banach space version)

Theorem

Let C, Y, Z be Banach spaces. Let U be a neighbourhood of $\mathbf{c_0}$ in C, V be a neighbourhood of $\mathbf{y_0}$ in Y and let $F : U \times V \to Z$ be a (Fréchet) C^1 mapping. Suppose that $F(\mathbf{c_0}, \mathbf{y_0}) = 0$ and that the linearization $d_y F(\mathbf{c_0}, \mathbf{y_0})(0, \cdot) : Y \to Z$ is an isomorphism. Then there is a neighbourhood $\tilde{U}$ of $\mathbf{c_0}$ in C, a neighbourhood $\tilde{V}$ of $\mathbf{y_0}$ in Y and a unique continuous mapping $G : \tilde{U} \to \tilde{V}$ such that $\mathbf{y_0} = G(\mathbf{c_0})$ and $F(\mathbf{c}, G(\mathbf{c})) = 0$ for all $x \in \tilde{U}$.

Example

- Consider the initial value problem

$$y'(x) - f(x, y) = 0, \quad y(0) = c, \quad x \in [0, 1].$$

  where $f(x, y) \in C^2(\mathbb{R}^2)$ with $f(x, 0) = 0$ for all $x \in [0, 1]$.
- This has the solution $y = y_0(x) = 0$ for $c = c_0 = 0$.
- Take $U = \mathbb{R}$, $V = C^1[0, 1]$, $Z = \mathbb{R} \times C^0[0, 1]$ and consider

$$F : U \times V \to Z : (c, y) \mapsto (y(0) - c, y'(x) - f(x, y)).$$

  Then $F(c, y) = 0$ iff y is a solution of the IVP with initial value c.
- A straightforward calculation of the linearization (Fréchet derivative) gives

$$d_y F(c_0, y_0)(0, v) = (v(0), v'(x) - \frac{\partial f}{\partial y}(x, 0)v(x)).$$

- By standard linear ODE theory, for every $(\xi, \eta(x)) \in Z$, there is a unique solution of the IVP

$$d_y F(c_0, y_0)(0, v) = (\xi, \eta(x)).$$

- Thus the linearization is an isomorphism, and so there an interval I containing $c_0 = 0$ and a continuous function

$$G : I \to C^1[0,1] : c \mapsto G(c) = y_c \in C^a[0,1]$$

  such that $F(c, y_c) = 0$.
- That is, $y = y_c(x)$ solves the IVP

$$y'(x) - f(x, y) = 0, \quad y(0) = c, \quad x \in [0,1].$$

Next steps

- Set up the EYM equations as a Banach space mapping with

$$\begin{aligned}\mathbf{y} &= (2m, S, \alpha, \omega, r^2\alpha', r^2\omega') \in C^1([0,x_*],\mathbb{R}^6)\\ \mathbf{c} &= \mathbf{y}(x_*) \in \mathbb{R}^6,\\ F(\mathbf{c},\mathbf{y}) &= (\mathbf{y}(x_*) - \mathbf{c}, \text{EYM equations}) \in \mathbb{R} \times C^0([0,x_*],\mathbb{R}^6).\end{aligned}$$

  where $x = r^{-1}$ and $x_* = (r_h + \delta)^{-1}$.

- The seed solution is the trivial solution of the EYM equations corresponding to S-AdS: $\mathbf{y_0} = \mathbf{c_0} = (2m_0, 1, 0, 1, 0, 0)$.
- **Lemma:** *There is a neighbourhood U of $\mathbf{c_0} \in \mathbb{R}^6$ and a neighbourhood V of $\mathbf{y_0} \in C^1([0,x_*],\mathbb{R}^6)$ such that for each $\mathbf{c} \in U$, there is a solution $\mathbf{y} \in V$ of the EYM equations on $[0,x_*]$ with $\mathbf{y}(x_*) = \mathbf{c}$. The neighbourhoods U and V may be chosen so that $\mu > 0$ and $S > 0$ throughout the interval $[0,x_*]$ and there is a unique $\mathbf{y} \in V$ for each $\mathbf{c} \in U$.*

Proposition

(Global existence of black hole solutions)
Let $r_h > 0, \Lambda < 0$ *and define* $\frac{1}{\ell^2} = -\frac{\Lambda}{3}$. *Then there is an open neighbourhood* Γ^{bh} *of* $\mathbf{g_0} = (1, 1, 0) \in \mathbb{R}^3$ *such that for each* $\mathbf{g} = (S_h, \omega_h, \alpha'_h) \in \Gamma^{\mathrm{bh}}$, *there exists a global solution of the EYM field equations defined on* $[r_h, +\infty)$ *with the event horizon asymptotic behaviour as* $r \to r_h^+$ *and the aAdS asymptotic behaviour as* $r \to \infty$. *These solutions are differentiable on* $[r_h, \infty)$ *(*C^1 *for* μ, S *and* C^2 *for* α, ω*) and satisfy* $\mu(r) > 0$ *for all* $r > r_h$. *Since* $\mu(r_h) = 0$, *the region* $r > r_h$ *corresponds to the exterior of a black hole.*

Comment

This requires some manifold theory to ensure that the local solutions on $[r_h, r_h + \delta]$ *are 'picked up' by the global solutions on* $[r_h + \delta, \infty)$.

Conclusions and further work

- Existence of dyonic BH in asymptotically AdS spacetime proven. Ditto dyonic solitons (dyons).
- Linear perturbation equations derived: operator theory approach to stability in static spherically symmetric spacetime seems to yield stability of some of these solutions.
- Future work: complete stability analysis - aAdS issues with evolution?
- Other contexts: higher dimensions, asymptotically de Sitter boundary conditions, physics of dyonic BH's.

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