Self-force in a modified radiation gauge for circular and eccentric orbits

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Motivation

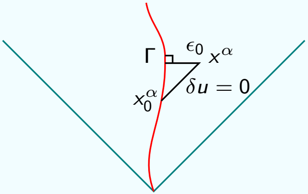
- One of the main sources of gravitational waves is the inspiral of compact objects into massive black holes in galactic nuclei.
- We work in the extreme mass-ratio inspiral (EMRI) regime, where the separation distance is small but the mass ratio of the bodies is large.
- The EMRI problem is amenable to a perturbative treatment, where the perturbation gives rise to the self-force (SF).
- Obtain accurate theoretical templates of EMRI waveforms. This waveforms have to include deviations from the geodesic motion due to the SF.
- Current calculations of the SF rely on numerical solutions of the linearised Einstein's equations in the Lorenz gauge. For Kerr the field equations in the Lorenz gauge are not separable.

- The treatment of black-hole perturbations for Kerr is much simpler in the radiation gauge, where it is possible to reconstruct the perturbations from the Weyl scalars.
- In the radiation gauge we don't have a SF formulation. The perturbation due to a point particle is a string-like 1-D singularity.
- We work in a gauge where it is "easy" to obtain the metric perturbations and relates through a regular gauge transformation to the Lorenz gauge. We call it *modified radiation gauge* (Mrad).
- The implementation will give the gravitational SF starting from a "force" in the outgoing radiation gauge. We use the mode-sum to obtain the SF.

$$F^{\alpha}_{self}(x_0) = \sum_{\ell=0}^{\infty} \left(F^{\alpha\ell}_{full\pm}(x_0) \mp A^{\alpha} L - B^{\alpha} - C^{\alpha}/L \right) - D^{\alpha}, \; (L \equiv \ell + 1/2).$$

SF in a modified radiation gauge

Consider a particle of mass **m** moving along Γ. Let the particle be embedded in the curvature of a massive Schwarzschild black hole of mass M.



In Mrad the perturbation near the particle has the same leading-order singularity as the Lorenz gauge,

$$h_{\alpha\beta}^{\mathrm{Mrad}} = 2\mu\epsilon_0^{-1}(g_{\alpha\beta} + 2u_\alpha u_\beta) + O(1).$$

We associate a given field point x^α with a "nearby" point on the worldline (δx^α). The most convenient choice is to take $x_0^\alpha(x)$ to be the point on Γ with the same retarded time as x^α ($\delta u = 0$).

The metric perturbation tensor transforms (from Rad→Mrad) according to

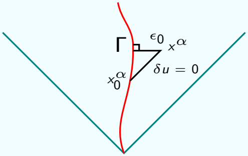
$$h^{\rm Mrad}_{\alpha\beta} = h^{\rm Rad}_{\alpha\beta} + \xi_{\alpha;\beta} + \xi_{\beta;\alpha}.$$

Which admits analytical solutions given by

$$\boxed{\xi^{\pm}_{\alpha} = \mp 2u_\alpha \ln(\epsilon_0 \mp u_\alpha \delta x^\alpha) + \frac{\delta_\alpha}{\Delta^{\pm}_{v}},}$$

where

$$\delta_\alpha \equiv 2\mathcal{L}\left\{0, -\frac{\delta\varphi}{u^u}, \frac{\delta\theta}{u^\varphi}, \frac{\delta\varphi}{u^\varphi}\right\}.$$



Before calculating the contributions to the SF we decompose $\xi_\alpha^\pm$ in ℓ-modes,

$$\xi^{\pm\ell}_{\alpha\perp} = \pm\delta^\ell_0\left(0,\ -\frac{\mathcal{L}^2 f_0}{r_0^2(\mathcal{E}-\dot{r})}, 0, \mathcal{L}\right) \quad \text{(in EF coordinates).}$$

We compare with the mode sum formula

$$F^{\rm Mrad}_\alpha = \sum_{\ell=0}^{\infty}\left[F^{{\rm Rad}\,\ell}_\alpha + \delta F^{{\rm Rad}\to{\rm Mrad}\,\ell}_\alpha - A_\alpha L - B_\alpha - C_\alpha/L\right] - D_\alpha,$$

due to the behaviour of the regularization parameters, we see that

$$\delta A_\alpha = \delta B_\alpha = \delta C_\alpha = 0, \quad \delta D_\alpha = \delta_\xi F^{{\rm Rad}\to{\rm Mrad}\,\ell=0}_\alpha.$$

Finally we calculate the change in the SF with

$$\delta F^{\alpha\,\ell}_{grav} = -\mathbf{m}\left[(g^{\alpha\lambda} + u^{\alpha}u^{\lambda})\frac{D^2\xi^{\ell}_{\lambda}}{D\tau^2} + R^{\alpha}_{\ \mu\lambda\nu}u^{\mu}\xi^{\ell\,\lambda}u^{\nu}\right].$$

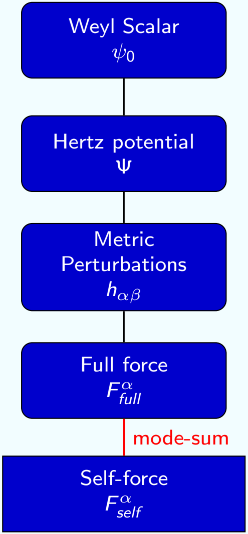
We obtain the explicit value of δD_{α}:

$$\delta D^{\pm}_{\alpha} = \left\{\pm\frac{\mathbf{m}^2\mathcal{L}^2 C_t(\mathcal{E},r,\dot{r})}{r^7(\mathcal{E}-\dot{r})^3}, \frac{\mathbf{m}^2\mathcal{L}^2 C_r(\mathcal{E},r,\dot{r})}{r^7 f(\mathcal{E}-\dot{r})^3}, 0, \pm\frac{2\mathbf{m}^2\mathcal{L} C_{\varphi}(\mathcal{E},r,\dot{r})}{r^4(\mathcal{E}-\dot{r})^2}\right\}.$$

For circular orbits they reduce to

$$\delta D^{\pm}_{\alpha} = \left\{0, \pm\frac{3\mathbf{m}^2 M^2}{r^{5/2}(r-3M)^{3/2}}, 0, 0\right\}.$$

Numerical Implementation

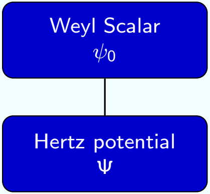


- Analytically solve for the $m = 0$ modes for $\ell > 2$.
- We integrate numerically the homogeneous Teukolsky equation (with $s = 2$) with ingoing boundary conditions for each ℓ, m.
- We obtain the corresponding Weyl curvature scalar ψ_0 at x_0^α by imposing junction conditions at x_0^α given by the source.

Weyl Scalar
ψ_0

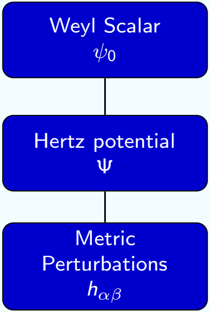
Obtained from Teukolsky equation for $s = 2$

$$(r^2 - 2Mr)\psi_0'' + 6(r - M)\psi_0' - \left[\frac{\omega^2 r^4}{r^2 - 2Mr} + \frac{4ir^2\omega(r - 3M)}{r^2 - 2Mr} - \bar{\eth}_3\eth_2\right]\psi_0 = -4\pi r^2 T_2.$$



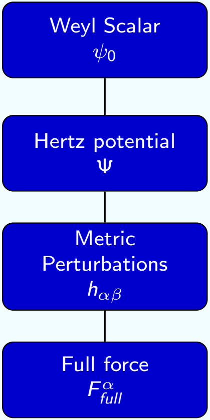
For circular orbits it can be obtained algebraically in terms of ψ_0

$$\Psi_{\ell m} = 8\frac{(-1)^m(\ell+2)(\ell+1)\ell(\ell-1)\bar{\psi}_{\ell,-m} + 12imM\Omega\psi_{\ell m}}{[(\ell+2)(\ell+1)\ell(\ell-1)]^2 + 144m^2M^2\Omega^2}.$$



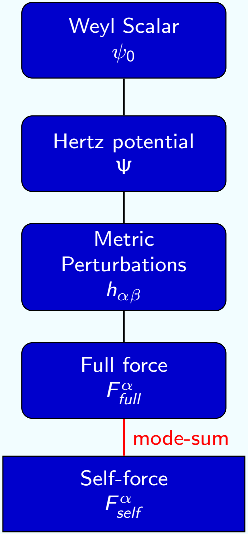
In terms of the Hertz potential

$$h_{\alpha\beta} = -\, r^4 \left\{ n_\alpha n_\beta (\bar{\boldsymbol{\delta}} + 2\beta)(\bar{\boldsymbol{\delta}} + 4\beta) + \bar{m}_\alpha \bar{m}_\beta (\boldsymbol{\Delta} + 5\mu - 2\gamma) \right.$$
$$(\boldsymbol{\Delta} + \mu - 4\gamma) - n_{(\alpha} \bar{m}_{\beta)} \left[(\bar{\boldsymbol{\delta}} + 4\beta)(\boldsymbol{\Delta} + \mu - 4\gamma) \right.$$
$$\left.\left. + (\boldsymbol{\Delta} + 4\mu - 4\gamma)(\bar{\boldsymbol{\delta}} + 4\beta) \right] \right\} \Psi + \text{c.c.}$$



The full force is obtained with the equation of motion

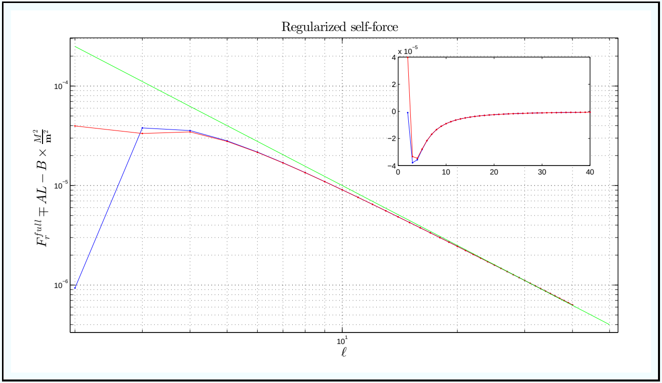
$$F^{\alpha}_{full} \equiv -\mathbf{m}(g^{\alpha\beta} + u^{\alpha}u^{\beta})\left(\nabla_{\mu}h_{\nu\beta} - \frac{1}{2}\nabla_{\beta}h_{\mu\nu}\right)u^{\mu}u^{\nu}.$$



We regularize each mode using the mode-sum formula:

$$F_{\alpha}^{\mathrm{Mrad}} = \sum_{\ell=0}^{\infty} \left[F_{\alpha}^{\mathrm{Rad}\,\ell} - A_{\alpha} L - B_{\alpha} \right] + \delta D_{\alpha}.$$

Self-force in ℓ-modes



ℓ-modes in log-log scale of the SF after regularization. Taken from the limit $r \to r_0^+$ (red) and the limit $r \to r_0^-$ (blue). The small graph is in linear scale.

Gauge invariant red-shift

Detweiler showed that for circular orbits in Schwarszchild there are two gauge invariant quantities that carry out non-trivial information about the conservative SF dynamics: Ω and $u^t \equiv U$. In practical calculations we compute:

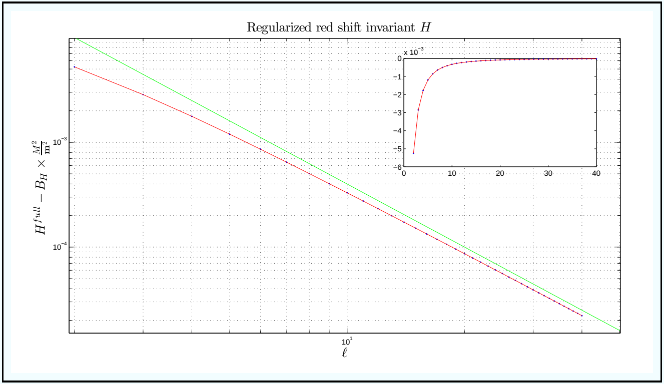
$$H \equiv \frac{1}{2} h^R_{\alpha\beta} u^\alpha u^\beta, \quad \frac{d\tau}{d\tilde{\tau}} = 1 + H,$$

where $\tilde{\tau}$ is the proper time along the geodesic of the effective metric $\tilde{g} = g + h^R$ and τ along the projection on g.

$$H^{\rm Mrad} = \sum_{\ell=0}^{\infty} \left[H^{{\rm Rad}\,\ell} - (B^H - \delta B^H) - (C^H - \delta C^H)/L \right] - (D^H - \delta D^H),$$

with $\delta B^H = \delta C^H = \delta D^H = 0$, for circular orbits.

H in ℓ-modes



ℓ-modes of H after regularization.

Summary and future work

- We have obtained the gauge transformation from the radiation gauge to the modified radiation gauge. This transformation has a null-string singularity (at each $\delta u = 0$) but it is possible to construct a regular solution along half-ray.
- The new mode-sum formula to obtain the GSF in a new modified radiation gauge (Schwarzschild).
- We have calculated numerically ℓ-modes contributions to SF and showed that the results from our implementation are consistent with all the regularization parameters given by the mode-sum formula.
- Include the low ℓ modes ($\ell = 0, 1$) to compute numerically the SF and the gauge invariant quantity H.
- Extend the numerical implementation to obtain the SF for non-circular orbits.
- Compute numerically the gravitational SF and the gauge invariant quantity H for the Kerr case.