Second Order Fermions and Chiral QED

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Outline

I. From first to second order fermions

II. Chiral QED (cQED)

III. Towards an understanding of gravity?

I. From first to second order fermions

Dirac fermions

We start with the Dirac Lagrangian coupled to Electrodynamics in $3+1$ dimensions with the metric $\eta_{\mu\nu} = (-,+,+,+)$. We have:

$$\mathcal{L}_D = -i\bar{\Psi}\not{D}\Psi - m\bar{\Psi}\Psi, \quad \not{D}\Psi = (\not{\partial} + ie\not{A})\Psi \tag{1}$$

with $\not{D} = \gamma^\mu D_\mu$.
The Lagrangian has the usual $U(1)$ gauge symmetry:

$$\Psi \ \mapsto \ e^{-ie\alpha(x)}\Psi, \quad A_\mu \ \mapsto \ A_\mu + \partial_\mu\alpha \tag{2}$$

Two-component fermions

Use the isomorphism

$$SO(1,3) \sim SL(2,\mathbb{C}) \tag{3}$$

With the map

$$\theta_\mu^{AA'} \;:\; x^\mu \mapsto x^{AA'} \equiv \theta_\mu^{AA'} x^\mu \tag{4}$$

where $\theta_\mu^{AA'}$ is called the soldering form (also referred to as "tetrad" or "vielbein").
It identifies Minkowski spacetime with the set of hermitian (2×2) matrices.

Two-component fermions

We can identify:

$$\Psi = \begin{pmatrix} \chi \\ \xi^\dagger \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\theta^\mu \\ \sqrt{2}\theta^\mu & 0 \end{pmatrix} \tag{5}$$

where we omit spinor indices.

- In order to go into a two-component fermions formalism $\rightarrow$ rewrite the Dirac Lagrangian in terms of these new quantities.

Two-component fermions

The Dirac Lagrangian then is:

$$\begin{aligned}\mathcal{L}_D &= -i\sqrt{2}\chi^\dagger\theta^\mu D_\mu\chi - i\sqrt{2}\xi^\dagger\theta^\mu D_\mu\xi - m(\chi\xi + \chi^\dagger\xi^\dagger) \\ &= -i\sqrt{2}\chi^\dagger_{A'}D^{A'A}\chi_A - i\sqrt{2}\xi^\dagger_{A'}D^{A'A}\xi_A - m(\chi^A\xi_A + \chi^\dagger_{A'}\xi^{\dagger A'})\end{aligned} \tag{6}$$

The electromagnetic $U(1)$ transformations are given by

$$\delta\xi = +ie\alpha\xi, \qquad \delta\chi = -ie\alpha\chi \tag{7}$$

$$D^{A'A}\xi_A = (\partial^{A'A} - ieA^{A'A})\xi_A, \qquad D^{A'A}\chi_A = (\partial^{A'A} + ieA^{A'A})\chi_A \tag{8}$$

Integrating one chirality out

Equations of motion

From the field equations for the primed spinors we get:

$$\xi^{\dagger A'} = -\frac{i\sqrt{2}}{m} D^{A'A}\chi_A, \qquad \chi^{\dagger A'} = -\frac{i\sqrt{2}}{m} D^{A'A}\xi_A. \tag{9}$$

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- Quadratic Lagrangian for the primed spinors
- Carry out the Berezin path-integral by effectively reinserting the EOM in the Lagrangian

Chiral QED

The Lagrangian for cQED is given by:

$$\mathcal{L}_{\text{chiral}} = -2D_{A'}{}^{A}\chi_A D^{A'B}\xi_B - m^2\chi^A\xi_A. \tag{10}$$

Together with the equations of motion:

$$2D_{A'}{}^{A}D^{A'B}\xi_B + m^2\xi^A = 0, \quad 2D_{A'}{}^{A}D^{A'B}\chi_B + m^2\chi^A = 0 \tag{11}$$

The unprimed spinor fields satisfy the generalised Klein-Gordon equation with non-zero bundle curvature.

Chiral QED

- The first order equations of motion are now seen as non-trivial reality conditions:

$$m\xi^{\dagger\, A'} = -i\sqrt{2}D^{A'A}\chi_A, \qquad m\chi^{\dagger\, A'} = -i\sqrt{2}D^{A'A}\xi_A \quad (12)$$

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- The fields are now normalised to have canonical mass dimension 1 (as seen from the Lagrangian).

II. Chiral QED (cQED)

Feynman rules

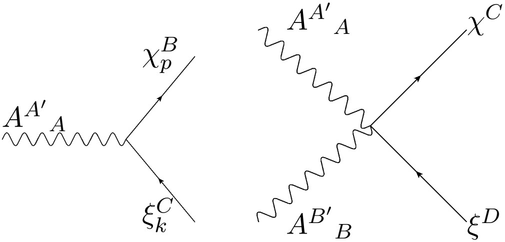
The propagator becomes very simple:

$$\langle 0|T\{\chi_A(p)\xi_B(-p)\}|0\rangle \equiv D(p)_{AB} = \frac{-i}{p^2+m^2}\epsilon_{AB} \qquad (13)$$

Feynman rules

We have two interaction vertices with Feynman rules (incoming momenta):

$$2ie\left[k_C{}^{A'}\epsilon_{BA} + p_B{}^{A'}\epsilon_{CA}\right], \qquad -2ie^2\epsilon^{A'B'}\epsilon_{AB}\epsilon_{CD} \tag{14}$$



Feynman rules

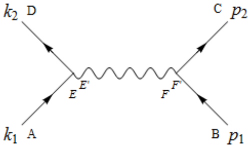
Computing Feynman amplitudes simply amounts to contracting spinors indices with the simple rules:

$$\lambda^A = \epsilon^{AB}\lambda_B, \quad \lambda_A = \lambda^B\epsilon_{BA} \tag{15}$$

No gamma matrices algebra to worry about!

$e^- \mu^- \to e^- \mu^-$ scattering

Simplest QED process: electron-muon scattering at tree level



Amputated amplitude $\mathcal{M}_{ABCD}$ for an incoming electron with momentum k_1 scattered off an incoming muon with momentum p_1. We have:

$$\mathcal{M}_{ABCD} = -\frac{4ie^2}{q^2}\Big[(k_1 \cdot p_1)_{AB}\, \epsilon_{CD} - (k_2 \cdot p_1)_{DB}\, \epsilon_{AC} - (k_1 \cdot p_2)_{AC}\, \epsilon_{BD} + (k_2 \cdot p_2)_{CD}\, \epsilon_{AB} \Big] \tag{16}$$

Unitarity

The Lagrangian

$$\mathcal{L}_{\text{chiral}} = -2D_{A'}{}^{A}\chi_A D^{A'B}\xi_B - m^2\chi^A\xi_A. \tag{17}$$

is obviously non-hermitian.

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- How can the theory be unitary?
- We have to impose reality conditions!
- In our case, we have a non-trivial real-structure that involves a derivative operator:

$$\dagger \;\mapsto\; \frac{i}{m}\mathcal{D}, \quad \left(\frac{i}{m}\mathcal{D}\right)^2 = I_V$$

Unitarity

- Imposed linearly (without any coupling to the gauge fields) on the free theory, this leads to a positive definite Hamiltonian $H_0 \sim a^\dagger a$.

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- It is checked perturbativaly, that unitarity holds. Moreover, the new vertex is essential to guarantee full non-linear reality.

III. Towards an understanding of gravity?

Unification and/or reformulation of gravity

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We expect this work to bring more insight into this issues in the future!

Thank you!